Complex Number

Geometrical Applications

1. Let $A(z_1)$, $B(z_2)$, $C(z_3)$, $D(z_4)$ in counterclockwise direction.

Vector BA = $z_1 - z_2$ Now turn vector BA with centre B through an angle α clockwisely.

$\therefore (z_1 - z_2)\text{ cis }(-\alpha)$ is a vector in the direction of the vector BC and of length |BA| = |AB|

Since the ratio of the length of BC to the length AB is k : 1,

$\therefore$ Vector BC = $k(z_1 - z_2)\text{ cis }(-\alpha) \Rightarrow z_3 - z_2 = k(z_1 - z_2)\text{ cis }(-\alpha)$

$\Rightarrow z_3 = z_2 + k(z_1 - z_2)\text{ cis }(-\alpha) = [z_2 + k(z_1 - z_2)\cos\alpha] - i\,k(z_1 - z_2)\sin\alpha$.

Vector CD = Vector BA $\Rightarrow z_4 - z_3 = z_1 - z_2 \Rightarrow z_4 = z_3 + z_1 - z_2$

$\Rightarrow z_4 = [z_2 + k(z_1 - z_2)\cos\alpha] - i\,k(z_1 - z_2)\sin\alpha + z_1 - z_2 = [z_1 + k(z_1 - z_2)\cos\alpha] - i\,k(z_1 - z_2)\sin\alpha$

2. Since z_2 describes a circle of radius a and centre at the origin,

$$\therefore\quad z_2 = a(\cos\theta + i\sin\theta), \qquad \frac{1}{z_2} = \frac{1}{a}(\cos\theta - i\sin\theta)$$

$$z_1 = x + yi = z_2 + \frac{1}{z_2} = a(\cos\theta + i\sin\theta) + \frac{1}{a}(\cos\theta - i\sin\theta) = \left(a + \frac{1}{a}\right)\cos\theta + i\left(a - \frac{1}{a}\right)\sin\theta$$

$$\therefore\quad x = \left(a + \frac{1}{a}\right)\cos\theta, \qquad y = \left(a - \frac{1}{a}\right)\sin\theta.$$ Eliminating θ, we get $$\frac{x^2}{(1+a^2)^2} + \frac{y^2}{(1-a^2)^2} = \frac{1}{a^2}.$$

3. Let the triangle with vertices be $z_k = x_k + y_k i$, k = 1,2,3.

Using the area formula in Cartesian Cordinates, the area

$$A = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2i}\begin{vmatrix} x_1 & y_1 i & 1 \\ x_2 & y_2 i & 1 \\ x_3 & y_3 i & 1 \end{vmatrix} = \frac{1}{2i}\begin{vmatrix} x_1 + y_1 i & y_1 i & 1 \\ x_2 + y_2 i & y_2 i & 1 \\ x_3 + y_3 i & y_3 i & 1 \end{vmatrix},$$ where C_1 is replaced by $C_1 + C_2$.

$$= -\frac{1}{4i}\begin{vmatrix} x_1 + y_1 i & -2y_1 i & 1 \\ x_2 + y_2 i & -2y_2 i & 1 \\ x_3 + y_3 i & -2y_3 i & 1 \end{vmatrix} = -\frac{1}{4i}\begin{vmatrix} x_1 + y_1 i & x_1 - y_1 i & 1 \\ x_2 + y_2 i & x_2 - y_2 i & 1 \\ x_3 + y_3 i & x_3 - y_3 i & 1 \end{vmatrix},$$ where C_2 is replaced by $C_1 + C_2$

$$A = \frac{i}{4}\begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ \bar{z}_1 & \bar{z}_2 & \bar{z}_3 \end{vmatrix}$$ Result follows.

4. Let $A(z_1)$, $B(z_2)$, $C(z_3)$. Then $z_1 - z_2$ is the vector BA .

Let $z = \dfrac{z_1 - z_2}{z_3 - z_2}$, Since $\text{Arg}(z) = \text{Arg}\dfrac{z_1 - z_2}{z_3 - z_2} = \text{Arg}(z_1 - z_2) - \text{Arg}(z_3 - z_2)$

therefore z is a complex number with $\text{Arg}(z) = \angle ABC$.

Since $\left|\dfrac{z_1 - z_2}{z_3 - z_2}\right| = \dfrac{|z_1 - z_2|}{|z_3 - z_2|}$, the modulus of z = $\dfrac{BA}{BC}$.

$$\frac{x-c}{b-c} = \frac{y-a}{c-a} \Rightarrow \begin{cases} \angle XCB = \angle YAC \\ \dfrac{CX}{CB} = \dfrac{AY}{AC} \end{cases} \Rightarrow \Delta BCX \sim \Delta CAY$$ (two sides in ratio, included $\angle$s equal)

Result follows later part of the given equality.

Let U, V be the centroids of ABC, XYZ and u , v are complex numbers representing U, V.

$$\frac{x-c}{b-c}=\frac{y-a}{c-a}=\frac{z-b}{a-b}=k \quad \Rightarrow x - c = k(b - c),\ y - a = k(c - a),\ z - b = k(a - b)$$

$$u=\frac{a+b+c}{3}, v=\frac{x+y+z}{3} \Rightarrow v-u=\frac{(x-c)+(y-a)+(z-b)}{3}=\frac{k(b-c)+k(c-a)+k(a-b)}{3}=0$$

$\therefore$ u = v and hence the centroids of ABC, XYZ are coincident.

5. **(i)** Let $A(z_1)$, $B(z_2)$, $C(z_3)$

$$\frac{z_3-z_1}{z_2-z_1}=\frac{z_1-z_2}{z_3-z_2} \quad \Leftrightarrow \quad z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

$\therefore$ $(z_1 - z_2)(z_2 - z_1) = (z_3 - z_1)(z_3 - z_2)$ and $(z_3 - z_1)(z_1 - z_3) = (z_2 - z_3)(z_2 - z_1)$

Combining we have $\frac{z_3-z_1}{z_2-z_1}=\frac{z_1-z_2}{z_3-z_2}=\frac{z_2-z_3}{z_1-z_3} \Rightarrow \operatorname{Arg}\left(\frac{z_3-z_1}{z_2-z_1}\right)=\operatorname{Arg}\left(\frac{z_1-z_2}{z_3-z_2}\right)=\operatorname{Arg}\left(\frac{z_2-z_3}{z_1-z_3}\right)$

Therefore $\angle A = \angle B = \angle C$.

(ii) The complex number $(z_2 - w_2)$ can be obtained from the complex number $(z_1 - w_2)$ by turning the latter through 120^o, hence

$x_3 - w_2 = (z_1 - w_2)\,\omega$, where $\omega = \text{cis } 120^o$. Note : $1+ \omega + \omega^2 = 0$, $\omega^3 = 1$.

$\Rightarrow w_2=\frac{z_3-z_1\omega}{1-\omega}$ and similarly $w_1=\frac{z_2-z_3\omega}{1-\omega}$, $w_3=\frac{z_1-z_2\omega}{1-\omega}$.

Since $w_1-w_2=\frac{z_1\omega+z_2-(1+\omega)z_3}{1-\omega}=\frac{z_1\omega+z_2+\omega^2z_3}{1-\omega}$,

$$w_2-w_3=\frac{z_2\omega+z_3-(1+\omega)z_1}{1-\omega}=\frac{z_2\omega+z_3+\omega^2z_1}{1-\omega},\ w_3-w_1=\frac{z_3\omega+z_1-(1+\omega)z_2}{1-\omega}=\frac{z_3\omega+z_1+\omega^2z_2}{1-\omega}$$

$$\Rightarrow \quad (w_1-w_2)(w_2-w_3)=\frac{(z_1\omega+z_2+\omega^2z_3)(z_2\omega+z_3+\omega^2z_1)}{(1-\omega)^2}$$

$$=\frac{\omega(z_1+\omega^2z_2+\omega z_3)\omega^2(z_1+\omega^2z_2+\omega z_3)}{(1-\omega)^2}=\left[\frac{z_1+\omega^2z_2+\omega z_3}{1-\omega}\right]^2=(w_3-w_1)^2$$

$\Rightarrow \frac{w_2-w_3}{w_1-z_3}=\frac{w_3-w_1}{w_2-w_1}$. By (i), $W_1W_2W_3$ is equilateral.

6. There are six possible different values the cross-ratio can take depending on the order in which the points z_i are given. Since there are 24 possible permutations of the four coordinates, some permutations must leave the cross-ratio unaltered. In fact, exchanging any two pairs of coordinates preserves the cross-ratio:

$$(z_1z_2, z_3z_4) = (z_2z_1, z_4z_3) = (z_3z_4, z_1z_2) = (z_4z_3, z_1z_2) = \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}=\lambda$$

The other five cross-ratios (with permutations) are (checking omitted):

$(z_1z_2,z_4z_3)=\frac{1}{\lambda}$, $(z_1z_3, z_2z_4) = 1-\lambda$, $(z_1z_3, z_4z_2) = \frac{1}{1-\lambda}$, $(z_1z_4, z_3z_2) = \frac{\lambda}{1-\lambda}$, $(z_1z_4, z_2z_3) = \frac{\lambda-1}{\lambda}$

Since $w_i-w_j=\frac{az_i+b}{cz_i+d}-\frac{az_j+b}{cz_j+d}=\frac{(ad-bc)(z_i-z_j)}{(cz_i+d)(cz_j+d)}$

$$(w_1w_2, w_3w_4) = \frac{(w_1-w_3)(w_2-w_4)}{(w_1-w_4)(w_2-w_3)}=\frac{\frac{(ad-bc)(z_1-z_3)}{(cz_1+d)(cz_3+d)}\times\frac{(ad-bc)(z_2-z_4)}{(cz_2+d)(cz_4+d)}}{\frac{(ad-bc)(z_1-z_4)}{(cz_1+d)(cz_4+d)}\times\frac{(ad-bc)(z_2-z_3)}{(cz_2+d)(cz_3+d)}}=\frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}=(z_1z_2, z_3z_4)$$

If $(z_1z_2, z_3z_4) = \dfrac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$ is real , then $\text{Arg}\dfrac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} = 0$ or π

$[\text{Arg}(z_1 - z_3) - \text{Arg}(z_1 - z_4)] - [\text{Arg}(z_2 - z_3) - \text{Arg}(z_2 - z_4)] = 0$ or π

$\angle Z_3Z_1Z_4 - \angle Z_3Z_2Z_4 = 0$ or π

If $\angle Z_3Z_1Z_4 - \angle Z_3Z_2Z_4 = 0$, then Z_1, Z_2, Z_3, Z_4 are concyclic. (Converse of $\angle$ in same segment)

If $\angle Z_3Z_1Z_4 - \angle Z_3Z_2Z_4 = \pi$, then Z_1, Z_2, Z_3, Z_4 are concyclic. (Opposite $\angle$s supplementary)

7. **(Method 1)**

For all points on the unit circle in the z-plane, $|z| = 1$ and any such point is given by $z = \cos\theta + i\sin\theta$. The corresponding points in the w-plane are given by:

$$|w| = \left|\frac{2z+i}{2-iz}\right| = \frac{|2\cos\theta + i(2\sin\theta+1)|}{|2+\sin\theta - i\cos\theta|} = \left[\frac{4\cos^2\theta + (2\sin\theta+1)^2}{(2+\sin\theta)^2 + \cos^2\theta}\right]^{1/2} = \left[\frac{4+4\sin\theta+1}{4+4\sin\theta+1}\right]^{1/2} = 1$$

and therefore lie on the unit circle in the w-plane.

(Method 2)

$$|w|^2 = w\overline{w} = \left(\frac{2z+i}{2-iz}\right)\left(\frac{2\overline{z}+i}{2-i\overline{z}}\right) = \frac{4z\overline{z}+1+2i(\overline{z}-z)}{4+z\overline{z}+2i(\overline{z}-z)} = 1, \qquad \text{since } z\overline{z} = |z|^2 = 1 .$$

8. **(a)** $$|f(z)|^2 = \left[\frac{a(z-b)}{1-z\overline{b}}\right]\left[\frac{\overline{a}(\overline{z}-\overline{b})}{1-\overline{z}b}\right] = \frac{a\overline{a}(z\overline{z}-z\overline{b}-\overline{z}b+b\overline{b})}{1-z\overline{b}-\overline{z}b+z\overline{z}b\overline{b}} = \frac{|a|^2(|z|^2-z\overline{b}-\overline{z}b+|b|^2)}{1-z\overline{b}-\overline{z}b+|z|^2|b|^2} \quad \ldots. \quad (1)$$

$$= \frac{1-z\overline{b}-\overline{z}b+|b|^2}{1-z\overline{b}-\overline{z}b+|b|^2} = 1 ,$$ since $|a| = |z| = 1$. Therefore, $|f(z)| = 1$.

(b) $|z| < 1$, $|b| < 1$ $\Rightarrow |z|^2 < 1$, $|b|^2 < 1$ $\Rightarrow (|z|^2 - 1)(|b|^2 - 1) > 0$ $\Rightarrow$ $|z|^2 + b|^2 < |z|^2 b|^2 + 1$

From (1), $$|f(z)|^2 = \frac{|a|^2(|z|^2+|b|^2-z\overline{b}-\overline{z}b)}{1-z\overline{b}-\overline{z}b+|z|^2|b|^2} < \frac{1(1+|z|^2|b|^2-z\overline{b}-\overline{z}b)}{1+|z|^2|b|^2-z\overline{b}-\overline{z}b} = 1 .$$

Since $|f(z)| > 0$, therefore $|f(z)| < 1$.

9. $(x + iy)(u + iv) = (xu - yv) + i(xv + yu) \quad \leftrightarrow \quad \begin{pmatrix} x & y \\ -y & x \end{pmatrix}\begin{pmatrix} u & v \\ -v & u \end{pmatrix} = \begin{pmatrix} xu-yv & xv+yu \\ -(xv+yu) & xu-yv \end{pmatrix}$

$i^2 = -1 \quad \leftrightarrow \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\alpha + y\sin\alpha \\ -x\sin\alpha + y\cos\alpha \end{pmatrix} \quad \ldots. \quad (1)$$

$$\begin{pmatrix} x' & y' \\ -y' & x' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} x & y \\ -y & x \end{pmatrix} = \begin{pmatrix} x\cos\alpha + y\sin\alpha & -x\sin\alpha + y\cos\alpha \\ -(-x\sin\alpha + y\cos\alpha) & x\cos\alpha + y\sin\alpha \end{pmatrix} \quad \ldots. \quad (2)$$

(1) shows the relationship when rectangular axes OX, OY are rotated counter-clockwise through an angle α to become OX’ , OY’.

(2) is equivalent to $x' + y'i = (x + yi)(\cos\alpha + i\sin\alpha)$ or $z' = e^{i\alpha} z$